

ASYMPTOTIC PROPERTIES OF ADAPTIVE ML ESTIMATORS IN LATENT VARIABLE MODELS

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Abstract: Latent variable models have been widely applied in different fields of research in which the constructs of interest are not directly observable, so that one or more latent variables are required to reduce the complexity of the data. In these cases, problems related to the integration of the likelihood function of the model arise since analytical solutions do not exist. In the recent literature, a numerical technique that has been extensively applied to estimate latent variable models is the adaptive Gauss-Hermite quadrature. It provides a good approximation of the function to be integrated, and it is more feasible than classical numerical techniques in presence of many latent variables and/or random effects. In this paper, we formally investigate the properties of maximum likelihood estimators based on adaptive quadratures used to perform inference in generalized linear latent variable models.

Key words and phrases: Gaussian quadrature, Generalized linear models, Laplace approximation, M-estimators.

1. Introduction

Models based on latent variables are used in many scientific fields, particularly in social sciences. For instance, in psychology, researchers often use concepts as intelligence and anxiety, that are difficult to observe directly, but that can be indirectly measured by surrogate data based on individual responses to a bat-

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tery of tests. In economics, welfare and poverty cannot be measured directly; hence income, expenditure and various other indicators on households are used as substitutes. Factor analysis is probably the most well-known latent variable model, based on the assumption of multivariate normality for the distribution of the manifest and latent variables. It has been extended by numerous researchers in order to deal with survey data that generally contain variables measured on binary, categorical or metric scales, or combinations of the above. Moustaki and Knott (2000) proposed a Generalized Linear Latent Variable Model (GLLVM) framework that allows the distribution of the manifest variables to belong to the exponential family, *i.e.* either continuous or discrete variables.

The purpose of GLLVM is to describe the relationship between a set of responses or items y_1, \dots, y_p , and a set of latent variables or factors z_1, \dots, z_q , that are fewer in number than the observed variables, but contain essentially the same information. The factors are supposed to account for the dependencies among the response variables in the sense that if the factors are held fixed, then the observed variables are independent. This is known as the assumption of conditional or local independence. The conditional distribution of $y_j|\mathbf{z}$ ($\mathbf{z} = [z_1, \dots, z_q]^T$) is taken from the exponential family (with canonical link functions)

$$g_j(y_j|\mathbf{z}) = \exp \left\{ \frac{y_j(\alpha_{0j} + \boldsymbol{\alpha}_j^T \mathbf{z}) - b_j(\alpha_{0j} + \boldsymbol{\alpha}_j^T \mathbf{z})}{\phi_j} + c_j(y_j, \phi_j) \right\} \quad j = 1, \dots, p,$$

where α_{0j} is the item-specific intercept, $\boldsymbol{\alpha}_j = [\alpha_{j1}, \dots, \alpha_{jq}]^T$ can be interpreted as factor loadings of the model, and ϕ_j is the scale parameter, that is of interest in the case of continuous observed components. The functions $b_j(\cdot)$ and $c_j(\cdot, \cdot)$ are known and assume different forms according to the different nature of y_j .

Under the assumption of conditional independence, the joint marginal distribution of the manifest variables is

$$f(\mathbf{y}, \boldsymbol{\theta}) = \int_{\mathbb{R}^q} g(\mathbf{y}|\mathbf{z})h(\mathbf{z})d\mathbf{z} = \int_{\mathbb{R}^q} \left[\prod_{j=1}^p g_j(y_j|\mathbf{z}) \right] h(\mathbf{z})d\mathbf{z} \quad (1.1)$$

with $\mathbf{y} = [y_1, \dots, y_p]^T$, $\boldsymbol{\theta} = [\alpha_{01}, \dots, \alpha_{0p}, \boldsymbol{\alpha}_1^T, \dots, \boldsymbol{\alpha}_p^T, \phi_1, \dots, \phi_p]^T$, and where \mathbf{z} is generally assumed to be multivariate standard normal, but the independence

assumption of the latent variables could be relaxed.

GLLVMs are designed as a flexible modelling approach. As a consequence, they are rather complex models, and their statistical analysis presents some difficulties, due to the fact that the latent variables are not observed. Maximum likelihood estimates in the GLLVM framework are typically obtained by using standard maximization algorithms, such as the EM and the Newton-Raphson algorithms (Moustaki and Knott, 2000; Huber et al., 2004). In both cases, the latent variables must be integrated out from the likelihood function, and numerical techniques have to be applied. Moustaki and Knott (2000) proposed the use of the Gauss-Hermite (GH) quadrature as a numerical approximation method. Although this is feasible in fairly simple models and tends to work well with moderate sample sizes, its application is often unfeasible when the number of latent variables increases. Moreover, GH can completely miss the maximum for certain functions and can be inefficient in other cases. To overcome these limitations, the Adaptive Gauss-Hermite (AGH) quadrature has become very popular in the latent variable literature. It allows to get a better approximation of the function to be integrated by adjusting the quadrature locations with specific features of the posterior density of the latent variables given the observations. Developed in the Bayesian context by Naylor and Smith (1982), it has been extended by several authors to deal with generalized linear mixed models. In particular, Schilling and Bock (2005) applied the AGH quadrature to approximate marginal likelihoods in IRT models with binary data, whereas Rabe-Hesketh et al. (2005) analyzed its behavior for generalized linear latent and mixed models. Furthermore, Joe (2008) compared the AGH with the Laplace approximation for a variety of discrete response mixed models. He found that the Laplace approximation becomes less adequate as the degree of discreteness increases and suggests to use AGH with binary and ordinal data. The prominent role of the AGH is also demonstrated by the fact that it is implemented in many statistical software used to fit GLLVM, such as in the function *gllamm* in STATA (Rabe-Hesketh and Skrondal, 2012), in MPLUS (Muthen and Muthen, 2010), and in the PROC NL MIXED in SAS (Lesaffre and Spiessens, 2001). However, to the best of our knowledge, inferential issues on the properties of the estimators based on the adaptive quadrature have not been addressed in the literature. In this paper, we formally investigate

these theoretical properties as function of both the sample size and the number of observed variables. Our results generalize those by Huber et al. (2004), who analyzed the properties of classical Laplace based estimators in GLLVM. Indeed, we show that the adaptive Gauss-Hermite quadratures share the same error rate of the higher (than one) order Laplace approximation.

The paper is organized as follows. In Section 2, we discuss the estimation of GLLVMs when the adaptive Gauss-Hermite quadrature is applied to approximate integrals. In Section 3, the relationship between AGH quadratures and the Laplace approximation is analyzed, and the asymptotic properties of the adaptive ML estimators are derived. A simulation study is implemented in Section 4 to analyze the finite sample properties of the estimators. Finally, in Section 5, a brief summary on the main findings of the paper is provided.

2. Estimation based on adaptive Gauss-Hermite quadrature

Maximum Likelihood (ML) estimates in the GLLVM framework are typically obtained by using either the EM or the Newton-Raphson algorithms. The key component for applying both the algorithms is the score vector of the observed data log-likelihood function. For a random sample of size n , the latter is defined as

$$\begin{aligned}\ell(\boldsymbol{\theta}) &= \sum_{l=1}^n \log f(\mathbf{y}_l, \boldsymbol{\theta}) \\ &= \sum_{l=1}^n \log \int_{\mathbb{R}^q} \prod_{j=1}^p \exp \left[\frac{y_{jl}(\alpha_{0j} + \boldsymbol{\alpha}_j^T \mathbf{z}_l) - b_j(\alpha_{0j} + \boldsymbol{\alpha}_j^T \mathbf{z}_l)}{\phi_j} + c_j(y_{jl}, \phi_j) \right] \\ &\quad (2\pi)^{-q/2} \exp \left[-\frac{1}{2} \mathbf{z}_l^T \mathbf{z}_l \right] d\mathbf{z}_l.\end{aligned}\tag{2.1}$$

It is easily shown that the score vector corresponding to expression (2.1) equals

$$S(\boldsymbol{\theta}) = \frac{\partial \ell(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \sum_{l=1}^n \frac{\partial}{\partial \boldsymbol{\theta}} \log f(\mathbf{y}_l, \boldsymbol{\theta})$$

$$\begin{aligned}
&= \sum_{l=1}^n \frac{1}{f(\mathbf{y}_l, \boldsymbol{\theta})} \int_{\mathbb{R}^q} \frac{\partial}{\partial \boldsymbol{\theta}} \log \{g(\mathbf{y}_l | \mathbf{z}_l) h(\mathbf{z}_l)\} d\mathbf{z}_l \\
&= \sum_{l=1}^n \frac{\int_{\mathbb{R}^q} S_l(\boldsymbol{\theta}; \mathbf{z}_l) g(\mathbf{y}_l | \mathbf{z}_l) h(\mathbf{z}_l) d\mathbf{z}_l}{\int_{\mathbb{R}^q} g(\mathbf{y}_l | \mathbf{z}_l) h(\mathbf{z}_l) d\mathbf{z}_l} \\
&= \sum_{l=1}^n \int_{\mathbb{R}^q} S_l(\boldsymbol{\theta}; \mathbf{z}_l) h(\mathbf{z}_l | \mathbf{y}_l) d\mathbf{z}_l = \sum_{l=1}^n E_{\mathbf{z}|\mathbf{y}}[S_l(\boldsymbol{\theta}, \mathbf{z}_l)] = E_{\mathbf{z}|\mathbf{y}}[S(\boldsymbol{\theta})]
\end{aligned} \tag{2.2}$$

where $S_l(\boldsymbol{\theta}; \mathbf{z}_l)$ denotes the complete-data score vector given by $\partial \log f(\mathbf{y}_l, \boldsymbol{\theta}) / \partial \boldsymbol{\theta} = \partial [\log g(\mathbf{y}_l | \mathbf{z}_l) + \log h(\mathbf{z}_l)] / \partial \boldsymbol{\theta}$. In words, the observed data score vector is expressed as the expected value of the complete-data vector with respect to the posterior distribution of the latent variables given the observations. This implies that eq. (2.2) plays a double role. If the score equations are solved with respect to $\boldsymbol{\theta}$, with $h(\mathbf{z}_l | \mathbf{y}_l)$ fixed at the $\boldsymbol{\theta}$ -value of the previous iteration, then this corresponds to the EM algorithm, whereas, if the score equations are solved with respect to $\boldsymbol{\theta}$ considering $h(\mathbf{z}_l | \mathbf{y}_l)$ also as a function of $\boldsymbol{\theta}$, then this corresponds to a direct maximization of the observed data log-likelihood $\ell(\boldsymbol{\theta})$. As we shall discuss further, based on this appealing feature, the estimators derived by applying either of these two algorithms will share the same theoretical properties.

Eq. (2.2) involves ratios of multidimensional integrals which cannot be solved analytically, except when all the $g_j(y_{jl} | \mathbf{z}_l)$ are normal. Consequently, an approximation of these integrals is needed, on which the bias and variance of resulting estimators will depend. In this paper, we study the properties of ML estimators based on the adaptive Gauss-Hermite approximation of integrals. This technique consists of adjusting the quadrature locations with specific features of the posterior density of the latent variables given the observations. This allows to get a better approximation of the function to be integrated. Naylor and Smith (1982) took the mean vector and covariance matrix of the normal density approximating the integrand to be the posterior mean and covariance matrix. Unfortunately, these posterior moments are not known exactly, but must themselves be obtained using adaptive quadratures. Integration is therefore iterative. To overcome this limitation, Liu and Pierce (1994) proposed an alternative procedure that consists in computing the mode of the integrand and its curvature (inverse of the Hessian

matrix) at the mode, so that numerical integration is avoided. In this case, the adaptive quadrature, when applied using one abscissa, is equivalent to the classical Laplace approximation, and its behavior has been analyzed in several papers on generalized linear models (Pinhero and Bates, 1995; Schilling and Bock, 2005; Skrondal and Rabe-Hesketh, 2004; Joe, 2008).

The application of the adaptive quadrature requires to rewrite eq. (1.1) as follows

$$f(\mathbf{y}_l, \boldsymbol{\theta}) = \int_{\mathbb{R}^q} \frac{g(\mathbf{y}_l|\mathbf{z}_l)h(\mathbf{z}_l)}{h_1(\mathbf{z}_l; \hat{\mathbf{z}}_l, \boldsymbol{\Psi}_l)} h_1(\mathbf{z}_l; \hat{\mathbf{z}}_l, \boldsymbol{\Psi}_l) d\mathbf{z}_l \quad (2.3)$$

where $h_1(\cdot; \hat{\mathbf{z}}_l, \boldsymbol{\Psi}_l)$ is a multivariate normal density with moments

$$\hat{\mathbf{z}}_l = \arg \max_{\mathbf{z}_l \in \mathbb{R}^q} [\log g(\mathbf{y}|\mathbf{z}_l) + \log h(\mathbf{z}_l)] \quad (2.4)$$

$$\boldsymbol{\Psi}_l = \left(-\frac{\partial^2 [\log g(\mathbf{y}|\mathbf{z}_l) + \log h(\mathbf{z}_l)]}{\partial \mathbf{z}_l^T \partial \mathbf{z}_l} \right) \bigg|_{\mathbf{z}_l = \hat{\mathbf{z}}_l}^{-1}. \quad (2.5)$$

A cartesian product rule based on the classical Gauss-Hermite quadrature is then applied so that the integrals have to be defined with respect to uncorrelated variables $\tilde{\mathbf{z}}_l$. Based on the Cholesky factorization of the covariance matrix $\boldsymbol{\Psi}_l = \mathbf{T}_l \mathbf{T}_l^T$, expression (2.3) can be rewritten as

$$f(\mathbf{y}_l, \boldsymbol{\theta}) = 2^{\frac{q}{2}} |\mathbf{T}_l| \int_{\mathbb{R}^q} g(\mathbf{y}_l | \sqrt{2} \mathbf{T}_l \tilde{\mathbf{z}}_l + \hat{\mathbf{z}}_l) h(\sqrt{2} \mathbf{T}_l \tilde{\mathbf{z}}_l + \hat{\mathbf{z}}_l) \exp[\tilde{\mathbf{z}}_l^T \tilde{\mathbf{z}}_l] \exp[-\tilde{\mathbf{z}}_l^T \tilde{\mathbf{z}}_l] d\tilde{\mathbf{z}}_l,$$

such that the AGH approximation of the density $f(\mathbf{y}_l, \boldsymbol{\theta}), l = 1, \dots, n$, is given by

$$\tilde{f}(\mathbf{y}_l, \boldsymbol{\theta}) = 2^{\frac{q}{2}} |\mathbf{T}_l| \sum_{t_1, \dots, t_q} g(\mathbf{y} | \mathbf{z}_{l,t_1, \dots, t_q}^*) h(\mathbf{z}_{l,t_1, \dots, t_q}^*) w_{t_1}^* \cdots w_{t_q}^* \quad (2.6)$$

where $\sum_{t_1, \dots, t_q} = \sum_{t_1=1}^k \cdots \sum_{t_q=1}^k$, being k the number of quadrature points selected for each latent variable, $\mathbf{z}_{l,t_1, \dots, t_q}^* = (z_{l,t_1}^*, \dots, z_{l,t_q}^*)^T = \sqrt{2} \mathbf{T}_l (z_{t_1}, \dots, z_{t_q})^T + \hat{\mathbf{z}}_l$ and $w_{t_k}^* = w_{t_k} \exp[z_{t_k}^2]$ are the AGH nodes and weights, respectively, with z_{t_k} being the classical GH nodes and $w_{t_k}, k = 1, \dots, q$, the corresponding weights.

From eq. (2.6), we obtain the approximated log-likelihood function

$$\begin{aligned} \tilde{\ell}(\boldsymbol{\theta}) &= \sum_{l=1}^n \log \left[2^{\frac{q}{2}} |\mathbf{T}_l| \sum_{t_1, \dots, t_q} \prod_{j=1}^p \exp \left(\frac{y_{jl} (\alpha_{0j} + \boldsymbol{\alpha}_j^T \mathbf{z}_{l,t_1, \dots, t_q}^*) - b_j (\alpha_{0j} + \boldsymbol{\alpha}_j^T \mathbf{z}_{l,t_1, \dots, t_q}^*)}{\phi_j} + c_j(y_{jl}, \phi_j) \right) \right. \\ &\quad \left. (2\pi)^{-q/2} \exp \left(-\frac{1}{2} \mathbf{z}_{l,t_1, \dots, t_q}^{*T} \mathbf{z}_{l,t_1, \dots, t_q}^* \right) w_{t_1}^* \cdots w_{t_q}^* \right]. \end{aligned} \quad (2.7)$$

The estimators of the model parameters are found by equating the corresponding derivatives of eq. (2.7) to zero, that is

$$\begin{aligned} \tilde{S}(\boldsymbol{\theta}) = \frac{\partial \tilde{\ell}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} &= \sum_{l=1}^n \frac{1}{\tilde{f}(\mathbf{y}_l, \boldsymbol{\theta})} \frac{\partial \tilde{f}(\mathbf{y}_l, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \\ &= \sum_{l=1}^n \frac{\sum_{t_1, \dots, t_q} S_l(\boldsymbol{\theta}, \mathbf{z}_{l,t_1, \dots, t_q}^*) g(\mathbf{y}_l | \mathbf{z}_{l,t_1, \dots, t_q}^*) h(\mathbf{z}_{l,t_1, \dots, t_q}^*) w_{t_1}^* \cdots w_{t_q}^*}{\sum_{t_1, \dots, t_q} g(\mathbf{y}_l | \mathbf{z}_{l,t_1, \dots, t_q}^*) h(\mathbf{z}_{l,t_1, \dots, t_q}^*) w_{t_1}^* \cdots w_{t_q}^*} \\ &= \sum_{l=1}^n \tilde{E}_{\mathbf{z}|\mathbf{y}} [S_l(\boldsymbol{\theta}, \mathbf{z}_l)] = \tilde{E}_{\mathbf{z}|\mathbf{y}} [S(\boldsymbol{\theta})] = 0 \end{aligned} \quad (2.8)$$

where, specifically,

$$\begin{aligned} S_l(\alpha_{0j}, \mathbf{z}_{l,t_1, \dots, t_q}^*) &= \frac{1}{\phi_j} \left[y_{jl} - \frac{\partial b_j (\alpha_{0j} + \boldsymbol{\alpha}_j^T \mathbf{z}_{l,t_1, \dots, t_q}^*)}{\partial \alpha_{0j}} \right] \\ S_l(\boldsymbol{\alpha}_j, \mathbf{z}_{l,t_1, \dots, t_q}^*) &= \frac{\mathbf{z}_{l,t_1, \dots, t_q}^*}{\phi_j} \left[y_{jl} - \frac{\partial b_j (\alpha_{0j} + \boldsymbol{\alpha}_j^T \mathbf{z}_{l,t_1, \dots, t_q}^*)}{\partial \boldsymbol{\alpha}_j} \right] \end{aligned}$$

and

$$S_l(\phi_j, \mathbf{z}_{l,t_1, \dots, t_q}^*) = -\frac{1}{\phi_j^2} \left[y_{jl} (\alpha_{0j} + \boldsymbol{\alpha}_j^T \mathbf{z}_{l,t_1, \dots, t_q}^*) - b_j (\alpha_{0j} + \boldsymbol{\alpha}_j^T \mathbf{z}_{l,t_1, \dots, t_q}^*) \right] + \frac{\partial c_j(y_{jl}, \phi_j)}{\partial \phi_j}.$$

Equations (2.8) provide a set of estimating equations defining the estimators for the model parameters. The same equations are derived in the E-step of the EM algorithm, in which the AGH quadrature is applied to approximate the E-step expectations (2.2). In the M-step, as in the direct maximization algorithm, improved estimates for the model parameters are obtained by maximizing the approximated expected score functions (2.8). For the scale parameter ϕ_j , closed form expressions can be derived, whereas, for the other parameters, a Newton

Raphson iterative scheme is used in order to solve the corresponding nonlinear maximum likelihood equations. In the derivation of the estimating equations, the model has been kept as general as possible without specifying the conditional distributions $g_j(y_j|\mathbf{z})$. In Appendix C, we give specific expressions for the quantity that are used in the likelihood function (2.1) and in the score functions (2.8) for binary manifest variables, whereas we refer to Bianconcini and Cagnone (2012) for count and categorical observed variables.

3. Statistical properties of the AGH-based estimators

To investigate the asymptotic properties of the maximum likelihood estimators based on the adaptive Gauss-Hermite quadrature, the error rate associated to the approximation (2.8) has to be determined. Liu and Pierce (1994) analyzed the asymptotic behavior of the AGH when it is used to approximate unidimensional integrals. Based on the fact that when applied with only one node it results in the Laplace approximation to the integral (De Bruijn, 1981; Barndorff-Nielsen and Cox, 1989), they proved that the adaptive quadrature based on k points can be alternatively thought as a higher (than one) order Laplace approximation. We now generalize this result to the multidimensional integral (1.1) as well as to the ratio of integrals (2.2), and we analyze the asymptotic accuracy of the corresponding Laplace approximations. The behavior of the latter for multidimensional integrals was studied by Barndorff-Nielsen and Cox (1989), Shun and McCullagh (1995), Shun (1997), and recently by Evangelou et al. (2011) for spatial generalized linear mixed models. Similarly, Raudenbush et al. (2000) considered improvements of the standard Laplace approximation obtained by incorporating higher order derivatives of the integrand.

For the derivations illustrated here, we follow the notation of Shun and McCullagh (1995) based on summation convention. Hence, an index that appears as a subscript and as a superscript implies a summation over all possible values of that index. We will denote the components of a vector sometimes by subscripts and sometimes by superscripts. The (i, j) th component of a matrix \mathbf{A} will be written as a_{ij} and its inverse (when exists) will have components a^{ij} . For any real function $f(\mathbf{z})$, $\mathbf{z} \in \mathbb{R}^q$, its derivative with respect to the i th component of \mathbf{z}

is denoted by a subscript, *i.e.* $f_i(\mathbf{z}) = \frac{\partial f(\mathbf{z})}{\partial z_i}$, $f_{ij}(\mathbf{z}) = \frac{\partial^2 f(\mathbf{z})}{\partial z_i \partial z_j}$, and, more generally, $f_{i_1, \dots, i_{2m}}(\mathbf{z}) = \frac{\partial^{2m} f(\mathbf{z})}{\partial z_{i_1} \dots \partial z_{i_{2m}}}$. In order to keep the notation as light as possible, we omit the individual subscript l .

3.1. Relationship with the Laplace approximation

The AGH quadrature implemented here is based on a tensor product of q univariate Gaussian quadratures based on the same number of quadrature points. In each dimension, the approximation (2.6) is exact for polynomials of degree $2k + 1$ or less. Hence, it provides a good approximation of the integral (1.1) if the ratio $\nu(\mathbf{z}) = \frac{g(\mathbf{y}|\mathbf{z})h(\mathbf{z})}{h_1(\mathbf{z}; \hat{\mathbf{z}}, \Psi)}$ can be approximated well by a q -variate polynomial, where the maximum exponent of all the monomials is at most $2k + 1$ (Tauchen and Hussey, 1991), in the region where the integrand is substantial. It follows that the effectiveness of the adaptive Gauss-Hermite approximation (2.6) can be evaluated by considering the Taylor series expansion of $\nu(\mathbf{z})$ around the mode $\hat{\mathbf{z}}$, that is

$$\nu(\mathbf{z}) = \nu(\hat{\mathbf{z}}) \left[1 + \sum_{m=3}^{\infty} \frac{1}{m!} c_{i_1, \dots, i_m}(\hat{\mathbf{z}}) (\mathbf{z} - \hat{\mathbf{z}})^{i_1, \dots, i_m} \right] \quad (3.1)$$

where (i_1, \dots, i_m) is a set of m indices, $c_{i_1, \dots, i_m}(\hat{\mathbf{z}}) = \frac{\nu_{i_1, \dots, i_m}(\hat{\mathbf{z}})}{\nu(\hat{\mathbf{z}})}$, $\nu_{i_1, \dots, i_m}(\hat{\mathbf{z}})$ denotes the partial derivatives of order m of ν with respect to z_{i_1}, \dots, z_{i_m} evaluated at the mode $\hat{\mathbf{z}}$, whereas $(\mathbf{z} - \hat{\mathbf{z}})^{i_1, \dots, i_m}$ refers to specific components of the vector $(\mathbf{z} - \hat{\mathbf{z}})$. The coefficients c_{i_1} and c_{i_1, i_2} are zero due to the choice of $h_1(\cdot; \hat{\mathbf{z}}, \Psi)$. Substituting the expansion (3.1) into the integral (1.1), we obtain the exact solution

$$f(\mathbf{y}, \boldsymbol{\theta}) = \nu(\hat{\mathbf{z}}) \left[1 + \sum_{m=2}^{\infty} \sum_Q \frac{1}{(2m)!} c_{i_1, \dots, i_{2m}}(\hat{\mathbf{z}}) \boldsymbol{\nu}^{q_1}(\hat{\mathbf{z}}) \dots \boldsymbol{\nu}^{q_m}(\hat{\mathbf{z}}) \right] \quad (3.2)$$

where the second sum is over the partition $Q = q_1 | \dots | q_m$ of $2m$ indices into m blocks, each of size 2, and $\boldsymbol{\nu}^{q_k}(\hat{\mathbf{z}})$, $k = 1, \dots, m$, are components of the covariance matrix Ψ . The Gauss-Hermite quadrature, for which k quadrature points are selected for each dimension, would be exact if the partial derivatives beyond the

$2(k+1)$ order in (3.2) are zero, that is

$$f(\mathbf{y}, \boldsymbol{\theta}) = \nu(\hat{\mathbf{z}}) \left[1 + \sum_{m=2}^k \sum_Q \frac{1}{(2m)!} c_{i_1, \dots, i_{2m}}(\hat{\mathbf{z}}) \nu^{q_1}(\hat{\mathbf{z}}) \cdots \nu^{q_m}(\hat{\mathbf{z}}) \right]. \quad (3.3)$$

To determine the asymptotic order of the approximation (3.3), its relationship with the higher order Laplace approximation of multidimensional integrals has to be taken into account. At this regard, the integral (1.1) has to be rewritten as

$$f(\mathbf{y}, \boldsymbol{\theta}) = \int_{\mathbb{R}_q} e^{[-L(\mathbf{z})]} d\mathbf{z} \quad (3.4)$$

where $L(\mathbf{z}) = -[\log g(\mathbf{y}|\mathbf{z}) + \log h(\mathbf{z})]$, such that $L(\mathbf{z}) = O(p)$. Assuming that $L(\mathbf{z})$ has a unique minimum $\hat{\mathbf{z}}$, Shun and McCullagh (1995) suggested the following expansion around that minimum

$$L(\mathbf{z}) = L(\hat{\mathbf{z}}) + \sum_{m=2}^{\infty} \frac{1}{m!} L_{i_1, \dots, i_m}(\hat{\mathbf{z}}) (\mathbf{z} - \hat{\mathbf{z}})^{i_1, \dots, i_m}$$

and applying the exponential function

$$e^{-L(\mathbf{z})} = (2\pi)^{q/2} |\boldsymbol{\Psi}|^{1/2} e^{-L(\hat{\mathbf{z}})} h_1(\mathbf{z}; \hat{\mathbf{z}}, \boldsymbol{\Psi}) \exp \left[\sum_{m=3}^{\infty} \frac{(-1)}{m!} L_{i_1, \dots, i_m}(\hat{\mathbf{z}}) (\mathbf{z} - \hat{\mathbf{z}})^{i_1, \dots, i_m} \right]$$

where $h_1(\cdot; \hat{\mathbf{z}}, \boldsymbol{\Psi})$ is a multivariate normal density with moments given in eq. (2.4) and (2.5). Based on exlog relations, the higher order term can be expressed as follows

$$\exp \left[\sum_{m=3}^{\infty} \frac{(-1)}{m!} L_{i_1, \dots, i_m}(\hat{\mathbf{z}}) (\mathbf{z} - \hat{\mathbf{z}})^{i_1, \dots, i_m} \right] = 1 - \sum_{m=3}^{\infty} \sum_P \frac{(-1)^t}{m!} L_{p_1}(\hat{\mathbf{z}}) \cdots L_{p_t}(\hat{\mathbf{z}}) (\mathbf{z} - \hat{\mathbf{z}})^{i_1, \dots, i_m}$$

such that the exact solution of the integral (3.4) is given by

$$(2\pi)^{\frac{q}{2}} |\boldsymbol{\Psi}|^{\frac{1}{2}} e^{-L(\hat{\mathbf{z}})} \left[1 - \sum_{m=2}^{\infty} \sum_{P, Q} \frac{(-1)^t}{(2m)!} L_{p_1}(\hat{\mathbf{z}}) \cdots L_{p_t}(\hat{\mathbf{z}}) L^{q_1}(\hat{\mathbf{z}}) \cdots L^{q_m}(\hat{\mathbf{z}}) \right] \quad (3.5)$$

where the second sum is over all partitions P, Q , such that $P = p_1 | \cdots | p_t$ is a partition of $2m$ indices into t blocks, each of size 3 or more, and $Q = q_1 | \cdots | q_m$

is a partition of $2m$ indices into m blocks, each of size 2. Each component $L^{q_k}(\hat{\mathbf{z}})$, $k = 1, \dots, m$, refers to specific elements of Ψ . As shown in the Appendix A, the exact solution (3.5) is equivalent to the one derived in eq. (3.2). It follows that the asymptotic order of the AGH approximation can be derived by truncating at $m = k$ the expansion (3.5), and by analyzing the asymptotic order associated to the bipartition (P, Q) related to $m = k + 1$. For fixed q , the usual asymptotic order of the term corresponding to the bipartition (P, Q) in (3.5) is $O(p^{t-m})$. It follows that the asymptotic error of the AGH based on k quadrature points is the same associated to the bipartition (P, Q) of $2(k + 1)$ indices, that is $O\left(p^{-\lceil \frac{k}{3} + 1 \rceil}\right)$ (see in Appendix A for more details).

It has to be noticed that when AGH quadratures are applied in the estimation of GLLVM, we need to approximate ratios of integrals as shown in eq. (2.2). The fully exponential solution (3.5) cannot be applied to the integral at the numerator, since the score functions $S(\boldsymbol{\theta}; \mathbf{z})$ are not necessarily positive. The integral has to be written in the standard form (Tierney et al., 1989; Evangelou et al., 2011)

$$\int_{\mathbb{R}^q} e^{-L(\mathbf{z})} S(\boldsymbol{\theta}; \mathbf{z}) d\mathbf{z}.$$

Beyond the Taylor series expansion of $L(\mathbf{z})$ around its minimum $\hat{\mathbf{z}}$, we have to consider a similar expansion of S around the same point, that is

$$S(\boldsymbol{\theta}; \mathbf{z}) = \sum_{m=0}^{\infty} S_{j_1, \dots, j_m}(\boldsymbol{\theta}; \hat{\mathbf{z}}) (\mathbf{z} - \hat{\mathbf{z}})^{j_1, \dots, j_m}.$$

Following Evangelou et al. (2011), it can be shown that

$$\begin{aligned} \int_{\mathbb{R}^q} S(\boldsymbol{\theta}; \mathbf{z}) g(\mathbf{y}|\mathbf{z}) h(\mathbf{z}) d\mathbf{z} &= (2\pi)^{\frac{q}{2}} |\Psi|^{\frac{1}{2}} e^{-[L(\hat{\mathbf{z}})]} \times \\ &\quad \left[\sum_{m=0}^{\infty} \sum_{s=0}^{2m} \sum_{P, Q} \frac{(-1)^t}{(2m)!} S_{j_1, \dots, j_s}(\boldsymbol{\theta}; \hat{\mathbf{z}}) L_{p_1}(\hat{\mathbf{z}}) \cdots L_{p_t}(\hat{\mathbf{z}}) L^{q_1}(\hat{\mathbf{z}}) \cdots L^{q_m}(\hat{\mathbf{z}}) \right] \end{aligned}$$

where P is a partition of $2m - s$ indices into t blocks, each of size 3 or more, and Q is a partition of the same indices together with $\{j_1, \dots, j_s\}$ into m blocks of size 2. Note that P and Q do not need to be connected. It follows that the exact Laplace solution of the expected score function (2.2) results

$$\frac{\sum_{m=0}^{\infty} \sum_{s=0}^{2m} \sum_{P,Q} \frac{(-1)^t}{2m!} S_{j_1, \dots, j_s}(\boldsymbol{\theta}; \hat{\mathbf{z}}) L_{p_1}(\hat{\mathbf{z}}) \cdots L_{p_t}(\hat{\mathbf{z}}) L^{q_1}(\hat{\mathbf{z}}) \cdots L^{q_m}(\hat{\mathbf{z}})}{\sum_{m=0}^{\infty} \sum_{P,Q} \frac{(-1)^t}{2m!} L_{p_1}(\hat{\mathbf{z}}) \cdots L_{p_t}(\hat{\mathbf{z}}) L^{q_1}(\hat{\mathbf{z}}) \cdots L^{q_m}(\hat{\mathbf{z}})}. \quad (3.6)$$

It will be perfectly account for the AGH approximation in eq. (2.8) if the partial derivatives, at both the numerator and denominator, of order greater than $2k$ ($\max m = k$) are zero. The corresponding Laplace approximation can be rewritten by regrouping in decreasing asymptotic order the scalar that appear in both the expansions, and by truncating the resulting series at an appropriate point. In symbols,

$$\frac{S(\boldsymbol{\theta}; \hat{\mathbf{z}}) + c_1^* p^{-1} + \cdots + c_r^* p^{-r} + \cdots + c_{[k/3]}^* p^{-[k/3]} + O\left(p^{-[\frac{k}{3}+1]}\right)}{1 + c_1 p^{-1} + \cdots + c_r p^{-r} + \cdots + c_{[k/3]} p^{-[k/3]} + O\left(p^{-[\frac{k}{3}+1]}\right)} \quad (3.7)$$

where the coefficients $c_r, r = 1, \dots, [\frac{k}{3}]$, are given by

$$c_r = \sum_{m=r+1}^{3r} \frac{(-1)^{m-r}}{(2m)!} L_{p_1}(\hat{\mathbf{z}}) \cdots L_{p_{m-r}}(\hat{\mathbf{z}}) L^{q_1}(\hat{\mathbf{z}}) \cdots L^{q_m}(\hat{\mathbf{z}})$$

with $p_1 | \cdots | p_t$ be a partition of $2m$ indices into $m - r$ blocks, each of size 3 or more, and $q_1 | \cdots | q_m$ is a partition of $2m$ indices into m blocks, each of size 2. On the other hand, the coefficients $c_r^*, r = 1, \dots, [\frac{k}{3}]$, results

$$c_r^* = \sum_{m=r}^{3r} \sum_{s=0}^{3r-m} \frac{(-1)^{m-r}}{(2m)!} S_{j_1, \dots, j_s}(\boldsymbol{\theta}; \hat{\mathbf{z}}) L_{p_1}(\hat{\mathbf{z}}) \cdots L_{p_{m-r}}(\hat{\mathbf{z}}) L^{q_1}(\hat{\mathbf{z}}) \cdots L^{q_m}(\hat{\mathbf{z}})$$

where $p_1 | \cdots | p_{m-r}$ is a partition of $2m - s$ indices into $m - r$ blocks, each of size 3 or more, and $q_1 | \cdots | q_m$ is a partition of the same indices together with $\{j_1, \dots, j_s\}$ into m blocks of size 2. Since $S_{j_1, \dots, j_s}(\boldsymbol{\theta}; \hat{\mathbf{z}}) = \frac{\partial L_{j_1, \dots, j_s}(\hat{\mathbf{z}})}{\partial \boldsymbol{\theta}}$, all the first derivatives of the score function will be zero due to the choice of $\hat{\mathbf{z}}$.

Based on long polynomial division, the approximated expected score functions equivalent to (2.8) are given by

$$\tilde{E}_{\mathbf{z}|\mathbf{y}}[S(\boldsymbol{\theta})] = \sum_{l=1}^n \left[S_l(\boldsymbol{\theta}, \hat{\mathbf{z}}_l) + c_1^{**} p^{-1} + \cdots + c_r^{**} p^{-r} + \cdots + O\left(p^{-[\frac{k}{3}+1]}\right) \right] \quad (3.8)$$

where the coefficients $c_r^{**}, r = 1, \dots, \left[\frac{k}{3}\right]$, can be determined as follows

$$c_r^{**} = [c_r^* - S(\boldsymbol{\theta}; \hat{\mathbf{z}})c_r] - c_{r-1}c_1^* - c_{r-2}c_2^* - \dots - c_1c_{r-1}^*$$

being

$$c_r^* - S(\boldsymbol{\theta}; \hat{\mathbf{z}})c_r = \sum_{m=r}^{3r} \sum_{s=2}^{3r-m} \frac{(-1)^{m-r}}{(2m)!} S_{j_1, \dots, j_s}(\boldsymbol{\theta}; \hat{\mathbf{z}}) L_{p_1}(\hat{\mathbf{z}}) \dots L_{p_{m-r}}(\hat{\mathbf{z}}) L^{q_1}(\hat{\mathbf{z}}) \dots L^{q_m}(\hat{\mathbf{z}})$$

and, in particular, $c_1^* - S(\boldsymbol{\theta}; \hat{\mathbf{z}})c_1 = \frac{1}{2} S_{j_1, j_2} L^{j_1, j_2}(\hat{\mathbf{z}})$.

3.2. Asymptotic behavior of the AGH-based estimators

To investigate the properties of the AGH approximated maximum likelihood estimators $\hat{\boldsymbol{\theta}}$, we analyze the asymptotic behavior of the corresponding Laplace-based estimators defined by (3.8). Our arguments are similar to those of Huber et al. (2004), who discussed classical Laplace estimators in GLLVM, and Rizopoulos et al. (2009) who analyzed the consistency of fully exponential Laplace estimators in joint models for survival and longitudinal data. At this regard, let $\boldsymbol{\theta}_0$ denote the true parameter value, then, under suitable regularity conditions, it is shown in the Appendix B that

$$(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) = O_p \left[\max \left(n^{-1/2}, p^{-\left[\frac{k}{3}+1\right]} \right) \right]. \quad (3.9)$$

Thus, $\hat{\boldsymbol{\theta}}$ is consistent as long as both n and p grow to ∞ . The $n^{-1/2}$ term comes from the standard asymptotic theory, whereas the $p^{-\left[\frac{k}{3}+1\right]}$ term derives from the AGH approximation. For $k \geq 3$, the AGH-based estimator is more accurate than the classical $O(p^{-1})$ Laplace-based estimators. Indeed, it shares the same accuracy of higher (than one) order Laplace estimators, but, with respect to this latter, the adaptive Gauss-Hermite is easier to be implemented, since it avoids derivative computations.

Based on the derivation of eq. (3.9) as presented in the Appendix B, we can deduce that, if $p = O(n^\rho)$ for $\rho > \frac{1}{\left[\frac{k}{3}+1\right]}$, then the AGH-based estimators will be asymptotically equivalent to the true maximum likelihood estimators that solve $S(\boldsymbol{\theta}) = 0$. However, in general, they are not maximum likelihood estimators

because of the approximation, but, as discussed by Huber et al. (2004) for classical Laplace estimators in GLLVM, they belong to the class of M -estimators. The latter are implicitly defined through a general Ψ -function as the solution in $\boldsymbol{\theta}$ of

$$\sum_{l=1}^n \Psi(\mathbf{y}_l; \boldsymbol{\theta}) = 0.$$

The Ψ -function for the AGH-based estimators are given by eq. (2.8). Then, under the conditions that are given by Huber (1981), the AGH-based estimators are asymptotically normal, *i.e.*

$$\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) \rightarrow^D MVN(\mathbf{0}, B(\boldsymbol{\theta}_0)^{-1} A(\boldsymbol{\theta}_0) [B(\boldsymbol{\theta}_0)^{-1}]^T)$$

as $n \rightarrow \infty$, where

$$\begin{aligned} A(\boldsymbol{\theta}_0) &= E[\Psi(\mathbf{y}_l; \boldsymbol{\theta}_0) \Psi^T(\mathbf{y}_l; \boldsymbol{\theta}_0)], \\ B(\boldsymbol{\theta}_0) &= -E \left[\frac{\partial \Psi(\mathbf{y}_l; \boldsymbol{\theta}_0)}{\partial \boldsymbol{\theta}} \right]. \end{aligned}$$

These conditions must be checked for the particular conditional distribution of each y_j (Huber et al., 2004).

4. Monte Carlo simulations

In this section, we investigate empirically the finite sample performance of the adaptive Gauss-Hermite and related Laplace-based estimators. We focus on latent variable models for binary data, since in this case the differences between numerical techniques should be better highlighted (Joe, 2008). We consider two simulation scenarios characterized by an increasing number of observed and latent variables. In particular, we generate data from a population that consists of six items satisfying a three factor model, and from a population based on ten observed variables that satisfy a five factor model. In both cases, the population parameters have been chosen in such a way that the item-specific intercepts and the factor loadings are drawn randomly from a log-normal distribution, with some loadings fixed to 0 to get unique solutions. For each scenario, 100 random samples have been considered with 200 subjects.

A crucial choice in the application of the AGH quadrature is the number of points needed to adequately approximate the likelihood function. In presence of binary data, Schilling and Bock (2005) suggested to select five quadrature points in presence of three latent factors, and three nodes for the five factor model. In both cases, the AGH shares the same asymptotic error of the Laplace approximation of order $O(p^{-2})$. For a different number of binary items (ranging from 8 to 15) observed over an increasing number of individuals (ranging from 50 to 1000), this relationship is illustrated in Figure (4.1). The latter shows the differences in the log-likelihood functions of the two- (left) and three- (right) factor models that are approximated using the AGH based on one (first row), three (second row), and five (third row) quadrature points, compared with the standard Laplace and the second order Laplace approximation. As expected, independently on the number of individuals, observed items and latent variables, when applied with just one quadrature node the AGH provides the same approximation as the standard Laplace technique. On the other hand, in this case, the second order Laplace approximation differs from the adaptive quadrature with higher differences for the model with three latent factors. The discrepancy between these two techniques drastically reduces as a larger number of nodes is used in the adaptive quadrature, independently on the number of factors, observed items and individuals. In the simulation study, we apply the adaptive quadrature using five and three quadrature points in the first and second scenario, respectively. Hence, its performance is compared with the second order Laplace approximation. The estimation is performed through the direct maximization algorithm described in Section 2, whose mathematical details for the case of binary observed items are provided in Appendix C. The algorithm is written in the statistical language R (R Development Core Team, 2010) and the program is available from the authors on request.

Table 4.1 illustrates the mean, bias, and Mean Square Error (MSE) of the parameter estimates obtained by applying both the techniques to the data generated by the three factor model. The results show that the AGH based on five quadrature points and the second order Laplace provide similar MSE values for all the model parameters. The Laplace seems to introduce a slightly larger bias in some estimates than the AGH, but a smaller variability with respect to the latter. On

the other hand, Table 4.2 shows the results for the five-factor models. In this specific case, the adaptive Gauss-Hermite has been applied with three quadrature points. We can notice that the results of the latter are very close to the one derived using the second order Laplace approximation, with similar conclusions to those drawn for the first scenario.

Table 4.1: Mean, bias and MSE of the parameter estimates for AGH based on 5 quadrature points and second order Laplace (*Lap2*) approximation in data generated by the three factor model.

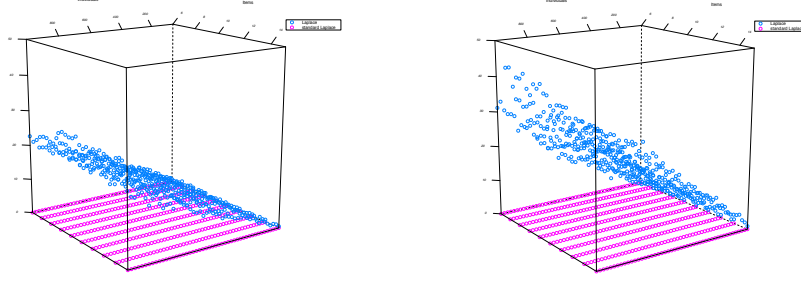
True	<i>AGH</i>			<i>Lap2</i>		
	Mean	Bias	MSE	Mean	Bias	MSE
$\alpha_{11} = 1.01$	0.72	-0.29	0.22	1.32	0.31	0.59
$\alpha_{21} = 0.91$	1.17	0.26	0.20	1.11	0.20	0.18
$\alpha_{31} = 0.50$	0.39	-0.12	0.11	0.35	-0.16	0.14
$\alpha_{41} = 0.74$	0.99	0.25	0.20	1.14	0.40	0.19
$\alpha_{51} = 1.16$	1.39	0.23	0.19	1.68	0.53	0.41
$\alpha_{61} = 1.22$	1.54	0.33	0.30	1.23	0.02	0.27
$\alpha_{12} = 0.00$	-	-	-	-	-	-
$\alpha_{22} = 0.83$	0.45	-0.38	0.24	0.21	-0.63	0.86
$\alpha_{32} = 0.44$	1.02	0.58	0.48	1.06	0.61	0.51
$\alpha_{42} = 0.88$	1.15	0.27	0.25	1.13	0.25	0.29
$\alpha_{52} = 1.73$	2.54	0.80	0.93	2.53	0.79	0.77
$\alpha_{62} = 1.46$	1.43	-0.03	0.27	1.36	-0.10	0.54
$\alpha_{13} = 0.00$	-	-	-	-	-	-
$\alpha_{23} = 0.00$	-	-	-	-	-	-
$\alpha_{33} = 1.45$	1.08	-0.37	0.33	1.21	-0.23	0.52
$\alpha_{43} = 1.05$	1.52	0.48	0.41	1.49	0.44	0.33
$\alpha_{53} = 0.62$	0.93	0.31	0.23	0.80	0.18	0.17
$\alpha_{63} = 0.91$	0.98	0.08	0.19	0.53	-0.38	0.26

5. Conclusions

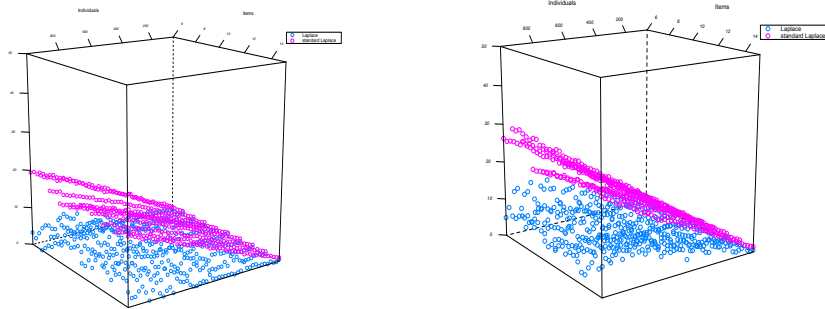
In this paper, we have investigated the theoretical properties of adaptive Gauss-Hermite based estimators in the GLLVM framework. Recently, the adaptive quadrature has played a prominent role in the latent variable model literature for approximating integrals defined over the latent space. It allows to overcome the main limitations of the commonly used techniques, such as the Gauss-Hermite quadrature and the standard Laplace approximation. Indeed, AGH is applicable to problems involving high-dimensional integrals where the former becomes impractical or computationally intensive, and it provides more accurate estimates than the latter, particularly when used for binary or ordinal

Figure 4.1: Differences in the log-likelihood functions approximated using AGH based on one/three/five nodes for each dimension, second order Laplace (blue) and standard Laplace (pink) approximations for two (left) and three (right) factor models.

One quadrature point for dimension



Three quadrature points for dimension



Five quadrature points for dimension

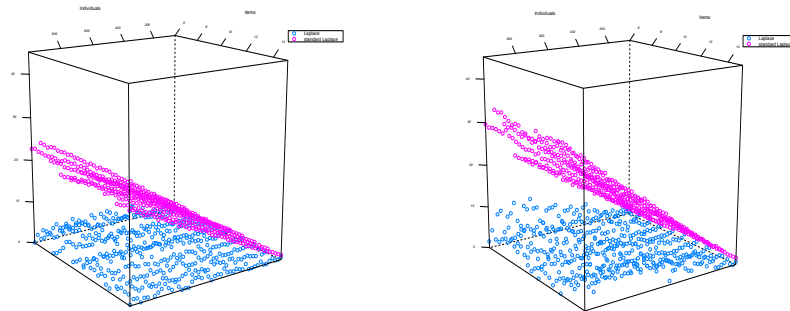


Table 4.2: Mean, bias and MSE of the parameter estimates for AGH based on 5 quadrature points, and second order Laplace (*Lap2*) approximation in the data generated by the five factor model.

True	<i>AGH</i>			<i>Lap2</i>		
	Mean	Bias	MSE	Mean	Bias	MSE
$\alpha_{11} = 1.01$	0.70	-0.31	0.21	1.28	0.27	0.28
$\alpha_{21} = 0.91$	1.27	0.36	0.20	1.33	0.42	0.44
$\alpha_{31} = 0.50$	0.87	0.37	0.30	0.57	0.06	0.05
$\alpha_{41} = 0.74$	1.14	0.40	0.31	0.85	0.11	0.12
$\alpha_{51} = 1.16$	1.83	0.67	0.52	1.98	0.82	0.90
$\alpha_{61} = 1.22$	1.22	0.00	0.15	0.66	-0.55	0.36
$\alpha_{71} = 0.55$	0.48	-0.07	0.10	0.59	0.04	0.05
$\alpha_{81} = 0.83$	1.10	0.27	0.19	0.90	0.07	0.07
$\alpha_{91} = 0.44$	1.01	0.57	0.40	1.05	0.61	0.42
$\alpha_{101} = 0.88$	1.05	0.17	0.12	1.17	0.29	0.11
$\alpha_{12} = 0.00$	-	-	-	-	-	-
$\alpha_{22} = 1.46$	1.39	-0.07	0.07	1.21	-0.25	0.25
$\alpha_{32} = 0.89$	0.59	-0.30	0.34	0.80	-0.08	0.15
$\alpha_{42} = 1.64$	1.27	-0.37	0.19	1.37	-0.27	0.22
$\alpha_{52} = 1.45$	0.60	-0.85	0.84	0.59	-0.86	0.96
$\alpha_{62} = 1.05$	0.92	-0.12	0.15	1.06	0.02	0.04
$\alpha_{72} = 0.62$	0.68	0.06	0.18	0.80	0.18	0.15
$\alpha_{82} = 0.91$	0.40	-0.50	0.36	0.19	-0.72	0.69
$\alpha_{92} = 1.59$	1.22	-0.36	0.23	2.02	0.43	0.54
$\alpha_{102} = 1.27$	0.95	-0.32	0.21	1.22	-0.05	0.07
$\alpha_{13} = 0.00$	-	-	-	-	-	-
$\alpha_{23} = 0.00$	-	-	-	-	-	-
$\alpha_{33} = 0.71$	1.10	0.38	0.35	1.13	0.42	0.43
$\alpha_{43} = 0.35$	1.02	0.67	0.54	1.01	0.66	0.46
$\alpha_{53} = 0.53$	1.46	0.93	0.94	1.46	0.93	0.90
$\alpha_{63} = 0.83$	0.97	0.15	0.18	0.64	-0.19	0.10
$\alpha_{73} = 0.71$	1.12	0.41	0.30	1.52	0.81	0.80
$\alpha_{83} = 0.65$	1.36	0.72	0.60	1.27	0.63	0.62
$\alpha_{93} = 0.95$	1.19	0.24	0.15	0.84	-0.11	0.04
$\alpha_{103} = 0.88$	1.23	0.35	0.27	0.68	-0.20	0.07
$\alpha_{14} = 0.00$	-	-	-	-	-	-
$\alpha_{24} = 0.00$	-	-	-	-	-	-
$\alpha_{34} = 0.00$	-	-	-	-	-	-
$\alpha_{44} = 1.10$	1.42	0.32	0.24	1.95	0.86	0.90
$\alpha_{54} = 0.50$	0.84	0.34	0.20	0.95	0.45	0.53
$\alpha_{64} = 0.49$	0.93	0.45	0.37	0.05	-0.44	0.31
$\alpha_{74} = 1.20$	0.67	-0.52	0.54	0.50	-0.70	0.54
$\alpha_{84} = 0.41$	0.43	0.02	0.13	0.11	-0.30	0.19
$\alpha_{94} = 0.85$	0.82	-0.03	0.14	0.43	-0.42	0.22
$\alpha_{104} = 0.72$	1.03	0.31	0.23	1.11	0.38	0.19
$\alpha_{15} = 0.00$	-	-	-	-	-	-
$\alpha_{25} = 0.00$	-	-	-	-	-	-
$\alpha_{35} = 0.00$	-	-	-	-	-	-
$\alpha_{45} = 0.00$	-	-	-	-	-	-
$\alpha_{55} = 0.62$	0.80	0.19	0.11	0.53	-0.09	0.13
$\alpha_{65} = 0.99$	1.32	0.34	0.22	1.23	0.24	0.45
$\alpha_{75} = 1.12$	1.04	-0.08	0.14	0.97	-0.16	0.15
$\alpha_{85} = 0.86$	1.05	0.19	0.20	1.42	0.56	0.59
$\alpha_{95} = 0.71$	0.67	-0.04	0.11	0.94	0.23	0.10
$\alpha_{105} = 1.39$	1.32	-0.07	0.13	1.73	0.34	0.41

data with small-sample sizes (Joe, 2008).

We have proved that, for multidimensional integrals, the AGH solution is asymptotically equivalent to the Laplace approximation that involves specific higher (than two) order derivatives of the integrand. Higher order Laplace approximations have been suggested in several papers on generalized linear models (Raudenbush et al., 2000; Evangelou et al., 2011; Bianconcini and Cagnone, 2012) as an alternative to classical methods for improving the accuracy of the estimates. This extension has been motivated by the well-known asymptotic properties that characterize the Laplace method, and by the fact that the approach does not suffer from the “curse of dimensionality”. However, the inclusion of higher order terms is computationally demanding as the order of the approximation increases. On the other hand, the AGH quadrature is easier to be implemented, but of course its computational complexity increases as the number of latent variables increases. Hence, AGH and higher order Laplace approximations can be seen as complementary approaches that share the same asymptotic properties.

We have shown that the AGH-based estimators are consistent as the sample size and number of observed variables grow to infinity. The convergence rate of these estimators depends also on the number of quadrature points used for each dimension. In general, these estimators are less efficient than maximum likelihood estimators because of the approximation, but belong to the class of M -estimators, for which the asymptotic properties are well-known such that correct inference can be performed.

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Appendix A: Asymptotic behavior of the multivariate AGH approximation

The higher order Laplace approximation of eq. (1.1) is derived by considering

$$f(\mathbf{y}, \boldsymbol{\theta}) = \int_{\mathbb{R}^q} e^{-L(\mathbf{z})} d\mathbf{z}$$

where $L(\mathbf{z}) = -[\log g(\mathbf{y}|\mathbf{z}) + \log h(\mathbf{z})]$, being $L(\mathbf{z}) = O(p)$. It is based on the Taylor series expansion of L around its minimum $\hat{\mathbf{z}}$, that is

$$L(\mathbf{z}) = L(\hat{\mathbf{z}}) + \frac{1}{2}L_{i_1, i_2}(\hat{\mathbf{z}})(\mathbf{z} - \hat{\mathbf{z}})^{i_1, i_2} + \sum_{m=3}^{\infty} \frac{1}{m!}L_{i_1, \dots, i_m}(\hat{\mathbf{z}})(\mathbf{z} - \hat{\mathbf{z}})^{i_1, \dots, i_m} \quad (1)$$

Substituting (1) into the integral, we obtain

$$\begin{aligned} \int_{\mathbb{R}^q} \exp[-L(\mathbf{z})] d\mathbf{z} &= (2\pi)^{\frac{q}{2}} |\boldsymbol{\Psi}|^{\frac{1}{2}} e^{-L(\hat{\mathbf{z}})} \int_{\mathbb{R}^q} h_1(\mathbf{z}; \hat{\mathbf{z}}, \boldsymbol{\Psi}) \exp \left[\sum_{m=3}^{\infty} \frac{(-1)^m}{m!} L_{i_1, \dots, i_m}(\hat{\mathbf{z}})(\mathbf{z} - \hat{\mathbf{z}})^{i_1, \dots, i_m} \right] d\mathbf{z} \\ &= (2\pi)^{\frac{q}{2}} |\boldsymbol{\Psi}|^{\frac{1}{2}} e^{-L(\hat{\mathbf{z}})} \left[1 - \sum_{m=2}^{\infty} \sum_{P, Q} \frac{(-1)^t}{(2m)!} L_{p_1}(\hat{\mathbf{z}}) \cdots L_{p_t}(\hat{\mathbf{z}}) L^{q_1}(\hat{\mathbf{z}}) \cdots L^{q_m}(\hat{\mathbf{z}}) \right]. \end{aligned}$$

where the second sum is over all partitions P, Q , such that $P = p_1 | \cdots | p_t$ is a partition of $2m$ indices into t blocks, each of size 3 or more, and $Q = q_1 | \cdots | q_m$ is a partition of $2m$ indices into m blocks, each of size 2. Each component $L^{q_k}(\hat{\mathbf{z}}), k = 1, \dots, m$, refers to specific elements of the covariance matrix $\boldsymbol{\Psi}$.

We want here to show that the exact higher order Laplace solution for the integral (1.1) is equivalent to the one based on the AGH quadrature given in eq. (3.2).

To do so, we need to show that

$$1 + \sum_{m=3}^{\infty} \sum_P (-1)^t L_{p_1}(\hat{\mathbf{z}}) \cdots L_{p_t}(\hat{\mathbf{z}}) (\mathbf{z} - \hat{\mathbf{z}})^{i_1, \dots, i_m} = \sum_{m=3}^{\infty} c_{i_1, \dots, i_m}(\hat{\mathbf{z}}) (\mathbf{z} - \hat{\mathbf{z}})^{i_1, \dots, i_m}. \quad (2)$$

At this regard, we can notice that, based on the exlog relations, the LHS term of eq. (2) is equal to $\exp \left[\sum_{m=3}^{\infty} L_{i_1, \dots, i_m}(\hat{\mathbf{z}}) (\mathbf{z} - \hat{\mathbf{z}})^{i_1, \dots, i_m} \right]$. This higher order term can be rewritten as

$$\pi(\mathbf{z}) = \exp \left[-L(\mathbf{z}) + L(\hat{\mathbf{z}}) + \frac{1}{2} L_{i_1, i_2}(\hat{\mathbf{z}}) (\mathbf{z} - \hat{\mathbf{z}})^{i_1, i_2} \right]$$

that it is equal to

$$\frac{(2\pi)^{-\frac{q}{2}} |\Psi|^{-\frac{1}{2}} g(\mathbf{y}|\mathbf{z}) h(\mathbf{z})}{g(\mathbf{y}|\hat{\mathbf{z}}) h(\hat{\mathbf{z}}) h_1(\mathbf{z}; \hat{\mathbf{z}}, \Psi)} = \frac{\nu(\mathbf{z})}{\nu(\hat{\mathbf{z}})} = c(\mathbf{z}).$$

Hence, the Taylor series expansion of $\pi(\mathbf{z})$ around the minimum $\hat{\mathbf{z}}$ can be written as

$$\begin{aligned} \pi(\mathbf{z}) &= 1 + \sum_{m=3}^{\infty} \frac{1}{m!} \pi_{i_1, \dots, i_m}(\hat{\mathbf{z}}) (\mathbf{z} - \hat{\mathbf{z}})^{i_1, \dots, i_m} \\ &= 1 + \sum_{m=3}^{\infty} \frac{1}{m!} c_{i_1, \dots, i_m}(\hat{\mathbf{z}}) (\mathbf{z} - \hat{\mathbf{z}})^{i_1, \dots, i_m} \end{aligned}$$

It follows that the AGH solution (3.2) and the Laplace one (3.5) are equivalent. Based on this relationship, it is possible to derive the asymptotic error associated with the AGH approximation (3.3) evaluating the equivalent Laplace approximation obtained by truncating (3.5) at $m = k$. Shun and McCullagh (1995) proved that, for fixed q , the usual asymptotic order of the term corresponding to the bipartition (P, Q) is $O(p^{t-m})$. The error rate of the AGH based on k quadrature points is the same associated to the bipartition (P, Q) of $2(k+1)$ indices in the expansion (3.5). In this case, the maximum number of blocks, each of size at least 3, for $2(k+1)$ indices is $\left\lceil \frac{2(k+1)}{3} \right\rceil$, where $\lceil r \rceil$ indicates the largest integer not exceeding r . Hence, being $m = k+1$, the AGH based on k quadrature points has associated asymptotic order equal to $O\left(p^{-\lceil \frac{k}{3} + 1 \rceil}\right)$.

Appendix B: Consistency of the AGH-based estimators

This section concerns with the consistency of the AGH based estimators. All the following proofs proceed along the lines of Vonesh (1996), who derived the rate of convergence of the estimator based on the classical Laplace approximation for nonlinear mixed effect models, and of Rizopoulos et al. (2009), who derived that rate for fully exponential Laplace based estimators in joint models for longitudinal and survival data. In particular, we work under the following assumptions:

1. $f(\mathbf{y}, \boldsymbol{\theta})$ is a well-defined density under the usual regularity conditions;
2. the true parameter value $\boldsymbol{\theta}_0$ is an interior point of the parameter space, and the Laplace estimator $\hat{\boldsymbol{\theta}}$ is an interior point in a neighborhood containing $\boldsymbol{\theta}_0$;
3. $\hat{\mathbf{z}} = \arg \max_{\mathbf{z} \in \mathbb{R}^q} [\log g(\mathbf{y}|\mathbf{z}) + \log h(\mathbf{z})]$ exists for all $i = 1, \dots, n$.

Let $\tilde{S}(\cdot)$ denote the approximated score vector according to the approximations (3.8); then we obtain

$$\begin{aligned} E_{\mathbf{z}|\mathbf{y}}[S(\hat{\boldsymbol{\theta}})] = S(\hat{\boldsymbol{\theta}}) &= \sum_{l=1}^n \left\{ S_l(\hat{\boldsymbol{\theta}}, \hat{\mathbf{z}}_l) + \dots + O\left(p^{-[\frac{k}{3}+1]}\right) \right\} \Rightarrow \\ n^{-1}S(\hat{\boldsymbol{\theta}}) &= n^{-1}\tilde{S}(\hat{\boldsymbol{\theta}}) + O\left(p^{-[\frac{k}{3}+1]}\right) \end{aligned} \quad (1)$$

since $\hat{\boldsymbol{\theta}}$ is chosen such that $\tilde{E}_{\mathbf{z}|\mathbf{y}}[S(\hat{\boldsymbol{\theta}})] = \tilde{S}(\hat{\boldsymbol{\theta}}) = 0$. Under the regularity conditions in assumption 1. and provided that $(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) = o_p(1)$, we can apply a Taylor series expansion in $S(\boldsymbol{\theta})$ around the true parameter vector $\boldsymbol{\theta}_0$:

$$S(\hat{\boldsymbol{\theta}}) = S(\boldsymbol{\theta}_0) + H(\boldsymbol{\theta}^*)(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0), \quad (2)$$

where $\boldsymbol{\theta}^*$ lies on the segment joining $\boldsymbol{\theta}_0$ and $\hat{\boldsymbol{\theta}}$, and $H(\boldsymbol{\theta}^*) = \frac{\partial S(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}^*} =$

$$\sum_{l=1}^n \frac{\partial S_l(\boldsymbol{\theta}, \hat{\mathbf{z}}_l)}{\partial \boldsymbol{\theta}} \Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}^*} = \sum_{l=1}^n H_l(\boldsymbol{\theta}^*, \hat{\mathbf{z}}_l).$$

From equations (1) and (2) we obtain

$$\begin{aligned} (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) &= - \left\{ n^{-1} \sum_{l=1}^n H_l(\boldsymbol{\theta}^*, \hat{\mathbf{z}}_l) \right\}^{-1} \left\{ n^{-1} [S(\boldsymbol{\theta}_0) - S(\hat{\boldsymbol{\theta}})] \right\} \Rightarrow \\ (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) &= - \left\{ n^{-1} \sum_{l=1}^n H_l(\boldsymbol{\theta}^*, \hat{\mathbf{z}}_l) \right\}^{-1} \left[n^{-1}S(\boldsymbol{\theta}_0) + O\left(p^{-[\frac{k}{3}+1]}\right) \right] \end{aligned}$$

In addition, under assumptions 1. and 2., we have that, as $n \rightarrow \infty$, $n^{-1}H(\boldsymbol{\theta}^*) \rightarrow^p E_{\mathbf{y}}[H(\boldsymbol{\theta}_0)]$, where the expectation is taken with respect to $f(\mathbf{y}, \boldsymbol{\theta})$, and $H(\boldsymbol{\theta}^*) = \sum_{l=1}^n H_l(\boldsymbol{\theta}^*, \hat{\mathbf{z}}_l)$. By further assuming that $E_{\mathbf{y}}\{H(\boldsymbol{\theta}_0)\}$ is non-singular, we obtain

$$\{n^{-1}H(\boldsymbol{\theta}^*)\}^{-1} \rightarrow^p E_{\mathbf{y}}\{H(\boldsymbol{\theta}_0)\}^{-1}$$

It follows that

$$\begin{aligned} (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) &= -E_{\mathbf{y}}[H(\boldsymbol{\theta}_0)]^{-1} \left[n^{-1}S(\boldsymbol{\theta}_0) + O\left(p^{-[\frac{k}{3}+1]}\right) \right] \\ &= O_p \left[\max \left(n^{-1/2}, p^{-[\frac{k}{3}+1]} \right) \right] \end{aligned}$$

where in the last step we use the fact that, under the regularity conditions 1., $n^{-1}S(\boldsymbol{\theta}_0) = O_p(n^{-1/2})$, and $E_{\mathbf{y}}\{H(\boldsymbol{\theta}_0)\} = O_p(1)$.

Appendix C: Technical details

Let $\mathbf{y} = (y_1, \dots, y_p)^T$ be a vector of observed binary variables, having a Bernoulli distribution with expectation $\pi_j(\mathbf{z}), j = 1, \dots, p$. Using the canonical link function for Bernoulli distribution we have

$$\pi_j(\mathbf{z}) = \frac{\exp(\alpha_{0j} + \boldsymbol{\alpha}_j^T \mathbf{z})}{1 + \exp(\alpha_{0j} + \boldsymbol{\alpha}_j^T \mathbf{z})}.$$

The scale parameter $\phi_j = 1$, such that the conditional distribution of each observed binary item given the latent variables \mathbf{z} is

$$g_j(y_j|\mathbf{z}) = \exp \left[y_j (\alpha_{0j} + \boldsymbol{\alpha}_j^T \mathbf{z}) + \log \left(\frac{1}{1 + \exp(\alpha_{0j} + \boldsymbol{\alpha}_j^T \mathbf{z})} \right) \right] \quad j = 1, \dots, p.$$

It follows that the approximate log-likelihood function (2.7) results

$$\begin{aligned} \tilde{\ell}(\boldsymbol{\theta}) &= \sum_{l=1}^n \log \left[2^{\frac{q}{2}} |\mathbf{T}_l| \sum_{t_1, \dots, t_q} \prod_{i=1}^p \exp \left(y_{ilt} (\alpha_{0j} + \boldsymbol{\alpha}_j^T \mathbf{z}_{l,t_1, \dots, t_q}^*) + \log \left(\frac{1}{1 + \exp(\alpha_{0j} + \boldsymbol{\alpha}_j^T \mathbf{z}_{l,t_1, \dots, t_q}^*)} \right) \right) \right. \\ &\quad \left. (2\pi)^{-q/2} \exp \left(-\frac{1}{2} \mathbf{z}_{l,t_1, \dots, t_q}^{*T} \mathbf{z}_{l,t_1, \dots, t_q}^* \right) w_{t_1}^* \cdots w_{t_q}^* \right]. \end{aligned}$$

where the AGH nodes and weights are derived by the classical Gauss-Hermite nodes z_{t_k} and weights w_{t_k} , $k = 1, \dots, q$, as follows

$$\mathbf{z}_{l,t_1,\dots,t_q}^* = (z_{l,t_1}^*, \dots, z_{l,t_q}^*)^T = \sqrt{2}\mathbf{T}_l(z_{t_1}, \dots, z_{t_q})^T + \hat{\mathbf{z}}_l$$

and

$$w_{t_k}^* = w_{t_k} \exp[z_{t_k}^2]$$

with \mathbf{T}_l derived by the Cholesky factorization of the matrix $\mathbf{\Psi}_l$, that is $\mathbf{\Psi}_l = \mathbf{T}_l \mathbf{T}_l^T$. The modes $\hat{\mathbf{z}}_l$ are obtained for each subject through the iterative scheme

$$\hat{\mathbf{z}}_l^{it+1} = \hat{\mathbf{z}}_l^{it} + \mathbf{\Psi}_l^{it} L(\hat{\mathbf{z}}_l^{it})$$

where “it” denotes the iteration counter,

$$L(\hat{\mathbf{z}}_l^{it}) = - \left. \frac{\partial [\log g(\mathbf{y}_l | \mathbf{z}_l) + \log h(\mathbf{z}_l)]}{\partial \mathbf{z}_l^T} \right|_{\mathbf{z}_l = \hat{\mathbf{z}}_l^{it}} = - \sum_{j=1}^p \boldsymbol{\alpha}_j \left[y_{jl} - \frac{\exp(\alpha_{0j} + \boldsymbol{\alpha}_j^T \hat{\mathbf{z}}_l^{it})}{1 + \exp(\alpha_{0j} + \boldsymbol{\alpha}_j^T \hat{\mathbf{z}}_l^{it})} \right] + \hat{\mathbf{z}}_l^{it},$$

and

$$\mathbf{\Psi}_l^{-1} = - \left. \frac{\partial^2 [\log g(\mathbf{y}_l | \mathbf{z}_l) + \log h(\mathbf{z}_l)]}{\partial \mathbf{z}_l^T \partial \mathbf{z}_l} \right|_{\mathbf{z}_l = \hat{\mathbf{z}}_l^{it}} = \sum_{j=1}^p \boldsymbol{\alpha}_j \boldsymbol{\alpha}_j^T \frac{\exp(\alpha_{0j} + \boldsymbol{\alpha}_j^T \hat{\mathbf{z}}_l^{it})}{1 + \exp(\alpha_{0j} + \boldsymbol{\alpha}_j^T \hat{\mathbf{z}}_l^{it})} + \mathbf{I}.$$

In this specific case, the complete data score functions (2.8) are given by

$$S_l(\alpha_{0j}, \mathbf{z}_{l,t_1,\dots,t_q}^*) = \frac{1}{\phi_j} \left[y_{jl} - \frac{\exp(\alpha_{0j} + \boldsymbol{\alpha}_j^T \mathbf{z}_{l,t_1,\dots,t_q}^*)}{1 + \exp(\alpha_{0j} + \boldsymbol{\alpha}_j^T \mathbf{z}_{l,t_1,\dots,t_q}^*)} \right]$$

$$S_l(\boldsymbol{\alpha}_j, \mathbf{z}_{l,t_1,\dots,t_q}^*) = \frac{\mathbf{z}_{l,t_1,\dots,t_q}^*}{\phi_j} \left[y_{jl} - \frac{\exp(\alpha_{0j} + \boldsymbol{\alpha}_j^T \mathbf{z}_{l,t_1,\dots,t_q}^*)}{1 + \exp(\alpha_{0j} + \boldsymbol{\alpha}_j^T \mathbf{z}_{l,t_1,\dots,t_q}^*)} \right].$$

The corresponding score equations have not closed form solutions, and a quasi-Newton procedure is used to solve implicit equations.

In the simulation study, the performance of the adaptive-based estimators has been compared with second order Laplace estimators. According to eq. (3.5), the latter have been derived by maximizing the following approximated log-likelihood

function

$$\tilde{\ell}(\boldsymbol{\theta}) = \sum_{l=1}^n \log \left[(2\pi)^{q/2} |\boldsymbol{\Psi}_l|^{1/2} \exp[-L(\hat{\mathbf{z}}_l)] (1 + c_1 p^{-1} + O(p^{-2})) \right]$$

where the individual modes $\hat{\mathbf{z}}_l$ are obtained through the iterative scheme defined above, and

$$c_1 = \sum_{m=2}^3 \frac{(-1)^{m-1}}{(2m)!} L_{p_1}(\hat{\mathbf{z}}_l) \cdots L_{p_{m-1}}(\hat{\mathbf{z}}_l) L^{q_1}(\hat{\mathbf{z}}_l) \cdots L^{q_m}(\hat{\mathbf{z}}_l)$$

with $p_1 | \cdots | p_{m-1}$ be a partition of $2m$ indices into $m-1$ blocks, each of size 3 or more, and $q_1 | \cdots | q_m$ is a partition of $2m$ indices into m blocks, each of size 2. In particular, following the notation by Raudenbush et al. (2000),

$$c_1 = -\frac{1}{8} \text{vec}^T[\boldsymbol{\Psi}_l \otimes \boldsymbol{\Psi}_l] \text{vec}[L^{(4)}(\hat{\mathbf{z}}_l)] + \frac{5}{24} \text{vec}^T[\boldsymbol{\Psi}_l \otimes \boldsymbol{\Psi}_l \otimes \boldsymbol{\Psi}_l] \text{vec}[L^{(3)}(\hat{\mathbf{z}}_l) \otimes L^{(3)}(\hat{\mathbf{z}}_l)]$$

where

$$L^{(3)}(\hat{\mathbf{z}}_l) = - \sum_{j=1}^p \text{vec}(\boldsymbol{\alpha}_j \boldsymbol{\alpha}_j^T) \boldsymbol{\alpha}_j^T \frac{\exp(\alpha_{0j} + \boldsymbol{\alpha}_j^T \hat{\mathbf{z}}_l) \left[1 - \exp(\alpha_{0j} + \boldsymbol{\alpha}_j^T \hat{\mathbf{z}}_l) \right]}{\left[1 + \exp(\alpha_{0j} + \boldsymbol{\alpha}_j^T \hat{\mathbf{z}}_l) \right]^3},$$

and

$$L^{(4)}(\hat{\mathbf{z}}_l) = - \sum_{j=1}^p \text{vec}[\text{vec}(\boldsymbol{\alpha}_j \boldsymbol{\alpha}_j^T) \boldsymbol{\alpha}_j^T] \boldsymbol{\alpha}_j^T \frac{\exp(\alpha_{0j} + \boldsymbol{\alpha}_j^T \hat{\mathbf{z}}_l) \left[1 - 4 \exp(\alpha_{0j} + \boldsymbol{\alpha}_j^T \hat{\mathbf{z}}_l) + \exp(\alpha_{0j} + \boldsymbol{\alpha}_j^T \hat{\mathbf{z}}_l)^2 \right]}{\left[1 + \exp(\alpha_{0j} + \boldsymbol{\alpha}_j^T \hat{\mathbf{z}}_l) \right]^4}.$$

As for the adaptive-based estimators, the score equations of both the intercepts and factor loadings have not closed form solutions, and a quasi-Newton procedure has been used to solve implicit equations.